One of the following statements is true and one is false. **SCORE: _____ / 18 PTS**

State clearly which statement is false, show that it is false, then write a **formal proof** for the true statement.

[a] The set of irrational numbers is closed under multiplication.

[b] If the sum of two integers is odd, then exactly one of the integers is odd.

YOU NEED TO SAY "FALSE" (2) AND PROVIDE THIS EXAMPLE TO GET ANY POINTS

[a] is false. For example, $\sqrt{2}$ is irrational, but $\sqrt{2}\times\sqrt{2}=2=\frac{2}{1}$ is not irrational.

[b] is true. There are two possible solutions, depending on whether you used contraposition or contradiction.

SOLUTION 1: ☆ GRADE AGAINST ONLY 1 SOLUTION

CONTRAPOSITIVE: For all integers x and y, if it is not the case that exactly one of x and y is odd, then $x+y$ is not odd.

PROOF BY CONTRAPOSITION:

(1)

Let x and y be particular but arbitrary chosen integers such that it is not the case that exactly one of x and y is odd.

So, either both x and y are odd, or neither x nor y are odd. (1)

CASE 1: Both x and y are odd

(1)

So, $x=2m+1$ and $y=2n+1$ for some integers m and n by definition of odd ($\frac{1}{2}$)

($\frac{1}{2}$) $x+y=2(m+n+1)$ where $m+n+1\in\mathbf{Z}$ by the closure of $\mathbf{Z}$ under $+$ (1)

($\frac{1}{2}$) So, $x+y$ is even by definition of even ($\frac{1}{2}$)

($\frac{1}{2}$) So, $x+y$ is not odd by Parity Property ($\frac{1}{2}$)

CASE 2: Neither x nor y are odd

($\frac{1}{2}$) ($\frac{1}{2}$)

So, both x and y are even by Parity Property

(1)

So, $x=2m$ and $y=2n$ for some integers m and n by definition of even ($\frac{1}{2}$)

($\frac{1}{2}$) $x+y=2(m+n)$ where $m+n\in\mathbf{Z}$ by the closure of $\mathbf{Z}$ under $+$ (1)

So, $x+y$ is even by definition of even ($\frac{1}{2}$)

($\frac{1}{2}$)

So, $x+y$ is not odd by Parity Property ($\frac{1}{2}$)

($\frac{1}{2}$)

(1) So, $x+y$ is not odd

Therefore, by contraposition, if the sum of two integers is odd, then exactly one of the integers is odd (1)

(1)

MUST STATE FULL SENTENCE FOR THIS POINT, NOT JUST "THE STATEMENT IS TRUE"

SOLUTION 2:

PROOF BY CONTRADICTION:

Suppose not, that is, suppose there are integers x and y such that $x+y$ is odd, $1\frac{1}{2}$

but it is not the case that exactly one of x and y is odd.

So, either both x and y are odd, or neither x nor y are odd. $1\frac{1}{2}$

CASE 1: Both x and y are odd

1

So, $x = 2m+1$ and $y = 2n+1$ for some $m, n \in \mathbf{Z}$ by definition of odd $\frac{1}{2}$

$\frac{1}{2}$ $x+y = 2(m+n+1)$ where $m+n+1 \in \mathbf{Z}$ by the closure of $\mathbf{Z}$ under $+$ 1

So, $x+y$ is even by definition of even $\frac{1}{2}$

$\frac{1}{2}$

CASE 2: Neither x nor y are odd

$\frac{1}{2}$

So, both x and y are even by Parity Property $\frac{1}{2}$

1 So, $x = 2m$ and $y = 2n$ for some $m, n \in \mathbf{Z}$ by definition of even $\frac{1}{2}$

$\frac{1}{2}$ $x+y = 2(m+n)$ where $m+n \in \mathbf{Z}$ by the closure of $\mathbf{Z}$ under $+$ 1

So, $\frac{1}{2}$ $x+y$ is even by definition of even $\frac{1}{2}$

2

So, $x+y$ is odd and $x+y$ is even (contradiction of Parity Property)

MUST STATE FULL SENTENCE FOR THIS POINT NOT JUST "THE STATEMENT IS TRUE"

Therefore, by contradiction, if the sum of two integers is odd, then exactly one of the integers is odd 1

1

Find the values of (-39) div 11 and (-39) mod 11. **SCORE: _____ / 4 PTS**

<u>Justify your answers VERY briefly. You do NOT need to write a proof.</u>

1 (-39) div $11 = -4$ and (-39) mod $11 = 5$ 1

since $-39 = -4 \times 11 + 5$ 2

Write the Quotient Remainder Theorem **symbolically**. **SCORE: _____ / 4 PTS**

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{1}{2}$ 1

$$\forall n \in \mathbf{Z}, \forall d \in \mathbf{Z}^+, \exists ! q, r \in \mathbf{Z} : n = dq + r \wedge 0 \le r < d$$

Fill in the blank with the value that makes the statement true, then write a **formal proof** of the resulting statement. **SCORE: _____ / 9 PTS**

"For all integers n, if $n \bmod 3 = 2$, then $(n^2 - 6) \bmod 3 = 1$."

ALL ITEMS 1 POINT UNLESS OTHERWISE NOTED

PROOF:

Let n be a particular but arbitrary chosen integer such that $n \bmod 3 = 2$.

So, $n = 3q + 2$ for some $q \in \mathbf{Z}$ by definition of mod.

2 So, $n^2 - 6 = 9q^2 + 12q - 2 = 3(3q^2 + 4q - 1) + 1$ where $3q^2 + 4q - 1 \in \mathbf{Z}$ by closure of $\mathbf{Z}$ under $\times$ and $+$.

So, $(n^2 - 6) \bmod 3 = 1$ by definition of mod.